

Renormalizable Tensor Field Theories*

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Extending tensor models at the field theoretical level, tensor field theories are nonlocal quantum field theories with Feynman graphs identified with simplicial complexes. They become relevant for addressing quantum topology and geometry in any dimension and therefore form an interesting class of models for studying quantum gravity. We review the class of perturbatively renormalizable tensor field theories and some of their features.

Keywords: Matrix models, tensor models, renormalization group, quantum geometry, quantum gravity.

1. Introduction

Introduced soon after the success of matrix models¹ and aiming at generalizing their success, tensor models² which address the weighted sum of simplicial manifolds in any dimension, turned out to be dramatically more involved than their lower dimensional cousins. We must recall that, among approaches for quantum gravity in low dimensions, matrix models¹ remain a prominent framework. Their achievement largely relies on the fact that geometry in 2D is of course well understood and on a fundamental tool which has given an handle on the partition function of matrix models: the 't Hooft large N expansion. Back in the 90's, much less is known about path integral of tensor models. Their phase transition and resulting geometries which turned out to be singular were investigated only through numerics. This approach needed a drastic change and rethinking. At the same period, Boulatov³ finds a link between a field theory formulation of tensor models and the Ponzano-Regge model for 3D gravity in the form of a lattice gauge field theory. The model by Boulatov introduces the concept of group holonomies in cellular complexes associated with Feynman graphs. It did not take long to see emerging a new framework called Group Field Theory (GFT).⁴

In 2010, Gurau discovered⁵ a large N expansion of a particular class of tensor models called colored.⁶ Then, a series of results followed among which the analytical proof that colored tensor models undergo a phase transition.⁷ After transition the type of geometries in the continuum limit was shown very singular: the so-called branched polymer geometries. This means that colored tensor models must be again

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enriched by other data to fulfill their goal to describe a large and smooth geometry in the continuum, a spacetime with all properties pertaining to it.

Simply because are quantum field theories, the question of the renormalization analysis in GFTs has been addressed in the meantime without a complete answer.⁸ In any renormalization program, there are difficulties of controlling the type divergences of amplitudes. This is the central question of finding a locality principle in field theory: to which type of interactions does correspond which type of propagator which will allow to perform the subtraction of any primitively divergent graph? For GFT models, this question was complex even for the simplest actions.

With the advent of colored tensor models, new types of effective interactions (integrating all colors except one^{9,10}) could have been studied under the renormalization group lens. A first tensor model in 4D, called Tensor Field Theory, was proved renormalizable at all orders.¹¹ In the following years, a wealth of renormalizable TFTs has been revealed. Opening a window on nonlocal QFTs, the renormalization analysis of TFTs has been performed in several dimensions (3 up to 6), using different background spaces (over Abelian, $U(1)$ and \mathbb{R} , and non Abelian $SU(2)$, direct spaces), and on models implementing the gauge constraint of GFT.^{12–19} See the review.²⁰

The study of renormalization covers more than a mathematical treatment for curing divergences of any QFT. It must explain the physics behind the model. In particular, the renormalization group (RG) flow analysis of TFTs should deliver at least hints for obtaining a classical spacetime at low energy in some parameter regime. To that extent, perturbative studies of the RG flow of TFTs have been undertaken with interesting results: many models turns out to be asymptotically free (AF).^{14–17,19} Asymptotic freedom has been a striking feature for the theory of Quantum Chromodynamics (QCD). Roughly speaking, at very high energy, an AF model flows toward a free and well-defined theory, whereas going in lower energy, the coupling constant of the model grows. This suggests a radical change of the theory involving a change of degrees of freedom. In QCD, the coupling of quarks increases which induces a binding, or confinement, of these particles which produces Hadrons, the most stable of which form the atom nucleus. For Tensorial Field Theory, asymptotic freedom becomes interesting mechanism indeed because we do not want to stay in a phase where the geometrical spacetime is apparently discrete and spanned by building blocks. The hope here is that, to draw a parallel with QCD, asymptotic freedom will induce a new phase for TFT models, towards new degrees of freedom able to generate a space with properties close to those of our spacetime.

The next section reviews the main ingredients to built a renormalizable TFT and section 3 gives a summary of our work and a future direction for investigations.

2. Tensor field theories

TFT models. Let $\phi_{\mathbf{P}}$ be a rank d complex tensor, where $\mathbf{P} = (p_1, p_2, \dots, p_d)$ a collection of indices. We denote $\bar{\phi}_{\mathbf{P}}$ its complex conjugate. In this work and for simplicity, we fix $p_k \in \mathbb{Z}$. A motivation on this choice is simple: introducing a complex function $\phi : U(1)^d \rightarrow \mathbb{C}$, $\phi_{\mathbf{P}}$ define nothing but the Fourier components of ϕ . In this way, a TFT, as described below, defines nothing but a particular field theory written in the momentum space of the torus. The physics here is given through the following *duality*: $\phi_{\mathbf{P}}$ is viewed as a $(d-1)$ -simplex.

A TFT model is defined via an action S built by convoluting copies of $\phi_{\mathbf{P}}$ and $\bar{\phi}_{\mathbf{P}}$ using kernels:

$$\begin{aligned} S[\bar{\phi}, \phi] &= \text{Tr}_2(\bar{\phi} \cdot K \cdot \phi) + \mu \text{Tr}_2(\phi^2) + S^{\text{int}}[\bar{\phi}, \phi], \\ \text{Tr}_2(\bar{\phi} \cdot K \cdot \phi) &= \sum_{\mathbf{P}, \mathbf{P}'} \bar{\phi}_{\mathbf{P}} K(\mathbf{P}; \mathbf{P}') \phi_{\mathbf{P}'}, \quad \text{Tr}_2(\phi^2) = \sum_{\mathbf{P}} \bar{\phi}_{\mathbf{P}} \phi_{\mathbf{P}}, \\ S^{\text{int}}[\bar{\phi}, \phi] &= \sum_{n_b} \lambda_{n_b} \text{Tr}_{n_b}(\bar{\phi}^{n_b} \cdot \mathcal{V}_{n_b} \cdot \phi^{n_b}), \end{aligned} \quad (1)$$

where Tr_{n_b} can be thought as generalized traces. Each of these expresses a type of convolution of the indices of the n_b couples of tensors according to a graphical pattern b (precisions on this will be given in a moment). The kernel K and \mathcal{V}_{n_b} are to be specified, μ is a mass and λ_{n_b} an interaction coupling constant. Setting \mathcal{V}_{n_b} to be of weight 1 kernel, Tr_{n_b} generate unitary invariants.⁹ Recalling that a rank d tensor is dual to a $(d-1)$ -simplex, the contraction of tensors forming an interaction represents a d -simplex obtained by gluing the $(d-1)$ -simplexes along their $(d-2)$ boundary simplexes.

Let us now specify the kinetic term in a rank d action:

$$K(\{p_i\}; \{p'_i\}) = \delta_{p_i, p'_i} \left(\sum_{i=1}^d p_i^{2a} \right), \quad \delta_{p_i, p'_i} := \prod_{i=1}^d \delta_{p_i, p'_i}, \quad \text{Tr}_2(\phi^2) = \sum_{p_i \in \mathbb{Z}} |\phi_{12\dots d}|^2, \quad (2)$$

where $a \in (0, 1]$, and where we use the notation $\phi_{12\dots d} := \phi_{p_1, p_2, \dots, p_d}$. The kernel K is the sum of $2a$ -power eigenvalues of d Laplacian operators acting over the d copies of $U(1)$. Thus, the dynamics of any model is quite standard. We focus now on interactions. To make clear the type of interactions we are considering, let us restrict to the rank $d = 3$, with a tensor $\phi_{123} := \phi_{p_1, p_2, p_3}$ (the general case can be inferred with no issue). To be even more specific, let us construct a ϕ^4 -like tensor field theory by convoluting four tensors. A first remarkable thing is that there is more than one way of convoluting four tensors. One of the possibilities is given by

$$\text{Tr}_{4;1}(\phi^4) = \sum_{p_i, p'_i} \phi_{123} \bar{\phi}_{1'23} \phi_{1'2'3'} \bar{\phi}_{12'3'}. \quad (3)$$

Hence, a ϕ^4 -theory may include several types of interactions and, naturally, $\phi^{2n>4}$ theories become far richer. The particular pattern of convolution of (3) will be explained graphically in the next paragraph discussing quantum aspects of the theory.

We now pass at the quantum level. From (2), we introduce a Gaussian field measure of covariance C of the form

$$d\nu_C(\phi, \bar{\phi}) = \prod_{\mathbf{P}} d\phi_{\mathbf{P}} d\bar{\phi}_{\mathbf{P}} e^{-\text{Tr}_2[\bar{\phi} \cdot (K + \mu) \cdot \phi]}, \quad C = 1/(K + \mu). \quad (4)$$

Also called propagator, $C(\mathbf{P}; \bar{\mathbf{P}})$ is represented graphically by a stranded line with d segments (see an example in rank 3 in Figure 1). Let us treat the interaction part of the theory. Tensor field interactions are represented by stranded vertex graphs. For instance, associated with (3), we obtain the vertex on the r.h.s of Figure 1. As one observes, a ϕ^{2n} interaction may be very well not symmetric with respect to its indices. At the level of the action S^{int} , we always sum over all colored symmetric terms to be able to renormalize a theory.

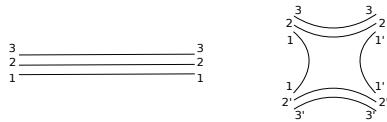


Fig. 1. A rank $d = 3$ propagator, as a stranded line (left), and the vertex $\text{Tr}_{4;1}(\phi^4)$.

As in any perturbative QFT, to study TFT correlators, we expand them at small coupling constants, use the Gaussian measure $d\nu_C$ (4) to generate Feynman graph amplitudes via the ordinary Wick theorem. A Feynman graph, in the present instance, has a specific stranded structure and represent a simplicial complex in dimension d . See an example in Figure 2. In the most generic case, summing over infinite degrees of freedom implies divergent amplitudes. In TFTs, divergences are localized in the graphs by the presence of loops called internal faces (see again, Figure 2).

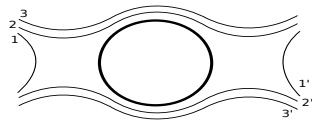


Fig. 2. An example of a rank $d = 3$ TFT graph and an internal face (put in bold) as a loop.

We consider now TFT with a field $\phi : (U(1)^D)^{\times d} \rightarrow \mathbb{C}$, producing a $D \times d$ field theory and seek conditions to obtain a regularized and renormalizable TFT. Note that the two parameters D and d will play a different role. The kinetic term with kernel K can be extended to $(U(1)^D)^{\times d}$.

Renormalizable TFTs. The renormalization is performed via a multiscale analysis.²¹ Such a program begins with a slice decomposition of the propagator as

$C = \sum_{i=0}^{\infty} C_i$ where each propagator in the slice i , namely C_i satisfies the upper bound $C_i \leq kM^{-2i}e^{-\delta M^{-i}(\sum_{s=1}^d |p_s|^a + \mu)}$ for some constants k , $M > 1$ and $i > 0$, and $C_0 \leq k$. Note that high i should select large momenta p_s of order $M^{\frac{i}{a}}$. We call this ultraviolet (UV) regime corresponding to short distances on $U(1)$. In opposite the regime, the slice $i = 0$ refers to the infrared (IR). The regularization scheme requires to introduce UV cut-off Λ on the sum over slices i and so that the regularized propagator is given by $C^\Lambda = \sum_{i=0}^{\Lambda} C_i$.

An amplitude associated with a graph $\mathcal{G}(\mathcal{V}, \mathcal{L})$ expresses, as in the usual way, as a product of propagator lines and vertex operators: $A_{\mathcal{G}} = \sum_{p_{s,v}} \prod_{\ell \in \mathcal{L}} C[\{\mathbf{P}_{v(\ell)}\}, \{\mathbf{P}'_{v'(\ell)}\}] \prod_{v \in \mathcal{V}; s} \delta_{p_{s,v}; p'_{s,v}}$. We perform a slice decomposition of all propagators, and collect the momentum scales $i_\ell \in [0, \Lambda]$ in a multi-index $m = (i_\ell)_{\ell \in \mathcal{L}}$ called momentum attribution. Then, we write $A_{\mathcal{G}} = \sum_m A_{\mathcal{G};m}$, where $A_{\mathcal{G};m}$ is the amplitude at fixed momentum attribution. The question is to provide the behavior of $A_{\mathcal{G};m}$ after an optimal integration of internal momenta in terms of the parameters M and of the set of the so-called quasi local subgraphs $\{G_k^i\}_{i,k}$.²¹

The following statement holds (power counting theorem):¹³ Let \mathcal{G} be a connected graph with set $\mathcal{L}(\mathcal{G})$ of lines of size $L(\mathcal{G})$, and set $\mathcal{F}_{\text{int}}(\mathcal{G})$ of internal faces of size $F_{\text{int}}(\mathcal{G})$, then there exists a constant $K_{\mathcal{G}}$ depending on the graph such that

$$|A_{\mathcal{G};m}| \leq K_{\mathcal{G}} \prod_{(i,k) \in \mathbb{N}^2} M^{\omega_d(G_k^i)}, \quad \omega_d(G_k^i) = -2aL(G_k^i) + D F_{\text{int}}(G_k^i). \quad (5)$$

The superficial divergence degree $\omega_d(\mathcal{G})$ of the graph \mathcal{G} determines if the amplitude associated with \mathcal{G} is divergent (when $\omega_d(\mathcal{G}) \geq 0$) or not.

As a second stage, we must treat the divergence degree and express the number of internal faces in terms of Gurau's degree of the underlying colored graph⁵ and of the degree of the boundary graph. The boundary graph encodes the boundary of the dual simplicial complex. As a definition, Gurau's degree is a sum of genera of canonical-colored surfaces of the TFT graph. It is proved that the amplitude is maximally divergent if it underlying graph has a vanishing degree. Studying the degree of divergence, it appears that the set of diverging graphs includes those with a vanishing degree, with a vanishing degree of their boundary graph and a restricted number of external fields. One obtains conditions on (a, D, d) and the maximal valence k_{max} in the vertex interactions yielding a renormalizable model. A subtraction scheme of the divergences can then be identified. The equations of the renormalized couplings in terms of the initial couplings define the so-called β -function equations which encode the renormalization group flow of the model. We obtain the table 1 of renormalizable models as well as their UV asymptotic behavior after calculation of their β -functions.¹³

3. Conclusion

We have identified a set of renormalizable actions built with tensor fields. At the UV-limit, many models turned out to be asymptotically free, in particular ϕ^4 mod-

Type	G_D	$\Phi^{k_{\max}}$	d	a	Renormalizability	UV behavior
TFT	$U(1)$	Φ^6	4	1	Just-	?
TFT	$U(1)$	Φ^4	3	$\frac{1}{2}$	Just-	AF
TFT	$U(1)$	Φ^6	3	$\frac{2}{3}$	Just-	?
TFT	$U(1)$	Φ^4	4	$\frac{3}{4}$	Just-	AF
TFT	$U(1)$	Φ^4	5	1	Just-	AF
TFT	$U(1)^2$	Φ^4	3	1	Just-	AF
TFT	$U(1)$	Φ^{2k}	3	1	Super-	AF

els. The UV-behavior of the ϕ^6 models is more subtle. The simplest ϕ^6 TFT model has been initially claimed AF¹⁵ but there are indications that this model could be actually safe in the UV.¹⁷ The existence of renormalizable tensor actions actually goes beyond the scope of the models presented in section 2 where TFTs appear in their simplest form, see for instance.¹⁶ For the lack of space, we cannot review these models in details.

Having understood their small coupling behavior, the study of TFTs has been recently pursued at the nonperturbative level through the Functional Renormalization Group approach.¹⁸ As a new result, the existence of an IR fixed point seems to be generic in TFTs. If confirmed, an IR fixed point also hints at a phase transition. In analogy with usual complex scalar field theory, the likely phases of TFT will be described by a spontaneous symmetry breaking mechanism. The two phases, the symmetric and broken one will correspond to positive and negative mass, respectively. The broken phase might be associated with a new condensed and geometrical ground state.²² This point deserves full investigations.

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